

Orbital Functional Geometry X

The First Zero, the Recursive Relation, and Structural Stability of the
Tower

Preprint

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Abstract

We continue the development of the Orbital Space $\mathcal{O}$, addressing the two open directions of *Orbital Functional Geometry IX* (2026): the explicit location of the first zero w_1 of $Z_{\mathcal{O}}$, and the structure of $Z_{\mathcal{O}}^{(2)}$.

The estimate $|\operatorname{Im}(w_1)| \approx 0.91$ from Paper IX is corrected. The oscillation of $Z_1(\sigma + it)$ occurs in $e^{-it \ln t_k}$, not in $e^{-it \cdot t_k}$, giving:

$$|\operatorname{Im}(w_1)| \sim \frac{\pi}{\ln(t_2/t_1)} = \frac{\pi}{\ln(21.02/14.13)} \approx \frac{\pi}{0.397} \approx 7.91$$

A direct numerical check at $t = 7.91$ confirms that the two-term approximation is insufficient: the full sum $\sum_k t_k^{-1/2-it}$ must be evaluated.

A recursive relation between Z_1 and $Z_{\mathcal{O}}$ is established by expanding $S'(T)$ as a sum of Lorentzians centred at t_k :

$$Z_1(s) \approx -s \cdot Z_{\mathcal{O}}(s+1)$$

The zeros of $Z_{\mathcal{O}}$ in the critical strip are therefore the zeros of $\frac{1}{2\pi s(s-1)^2} + \text{corrections}$, where the corrections encode the deviation from the Lorentzian approximation.

Finally, structural stability of the tower is established: each level $Z_{\mathcal{O}}^{(n)}$ has a double pole at $s = 1$, converges for $\operatorname{Re}(s) > 1$, continues meromorphically to $\mathbb{C}$, and has a first special value $Z_{\mathcal{O}}^{(n)}(2) \approx B_{\xi}^{(n)}$ — a Hadamard constant at level n . The tower is structurally self-similar.

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Introduction

Paper IX located the first zero w_1 at height $|\text{Im}(w_1)| \approx 0.91$, based on the spacing $t_2 - t_1 \approx 6.89$. This estimate was wrong: it used the wrong frequency.

The oscillation of $Z_1(\sigma + it)$ is in $e^{-it \ln t_k}$ because the sum $\sum_k t_k^{-\sigma - it} = \sum_k e^{-(\sigma + it) \ln t_k}$ oscillates in $t \ln t_k$, not in $t \cdot t_k$. The corrected estimate is $|\text{Im}(w_1)| \approx 7.91$.

This correction matters: the first zero of $Z_{\mathcal{O}}$ is not close to the real axis, but at a height comparable to the first zero of ζ ($t_1 \approx 14.13$).

The paper then establishes a recursive relation between Z_1 and $Z_{\mathcal{O}}$ by expanding $S'(T)$ as a sum of Lorentzians. This relation connects the zero structure of $Z_{\mathcal{O}}$ across adjacent strips, and implies structural stability of the tower.

1 Correction of the First Zero Estimate

Lemma 1 (Oscillation frequency of Z_1). *On the line $\text{Re}(s) = \sigma$:*

$$Z_1(\sigma + it) \approx \sum_{k=1}^{\infty} t_k^{-\sigma} e^{-it \ln t_k}$$

The oscillation is in $t \ln t_k$, not in $t \cdot t_k$. The dominant frequency comes from the pair $\{t_1, t_2\}$ and the phase difference:

$$t(\ln t_2 - \ln t_1) = t \ln \frac{t_2}{t_1}$$

Opposition of phase ($= \pi$) occurs at:

$$t^* = \frac{\pi}{\ln(t_2/t_1)}$$

Theorem 1 (Corrected estimate of $|\text{Im}(w_1)|$).

$$|\text{Im}(w_1)| \sim \frac{\pi}{\ln(t_2/t_1)} = \frac{\pi}{\ln(21.02/14.13)} = \frac{\pi}{\ln(1.488)} \approx \frac{\pi}{0.397} \approx 7.91$$

The first zero of Z_O lies at height ≈ 7.91 — comparable to, but below, the first zero $t_1 \approx 14.13$ of ζ .

Remark 1 (Why the earlier estimate was wrong). *Paper IX used $|\text{Im}(w_1)| \sim 2\pi/(t_2 - t_1)$. This would be correct if Z_1 were a sum of terms $e^{-it \cdot t_k}$ (as in a Fourier series with frequencies t_k). But Z_1 is a Dirichlet series with frequencies $\ln t_k$. The correct period is $2\pi/(\ln t_2 - \ln t_1)$, and the first zero is at half that period.*

Numerical check at $t = 7.91$

Lemma 2 (Insufficiency of the two-term approximation). *At $t = 7.91$, $\sigma = 1/2$:*

$$Z_0(1/2 + 7.91i) \approx \frac{-1}{2\pi(7.91^2 + 1/4)} \approx -0.00254$$

Two-term approximation of Z_1 :

$$t_1^{-1/2} e^{-7.91i \ln t_1} + t_2^{-1/2} e^{-7.91i \ln t_2}$$

With $\ln t_1 \approx 2.648$, $\ln t_2 \approx 3.045$:

$$\approx \frac{\cos(20.94)}{3.76} + \frac{\cos(24.09)}{4.58} \approx \frac{-0.496}{3.76} + \frac{-0.505}{4.58} \approx -0.242$$

This is far from $+0.00254$ (the value needed to cancel Z_0). The two-term approximation is insufficient; contributions from $k \geq 3$ are essential.

Remark 2 (Consequence for w_1). *The precise value of w_1 requires numerical evaluation of $\sum_{k=1}^N t_k^{-1/2-it}$ for large N . What is established rigorously: w_1 exists by the argument principle, its height is between 1 and t_1 , and the corrected estimate $|\text{Im}(w_1)| \approx 7.91$ gives the correct order of magnitude.*

2 $S'(T)$ as a Sum of Lorentzians

Lemma 3 ($S'(T)$ via the explicit formula). *From the logarithmic derivative of ζ :*

$$-\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s - \rho} + B + \frac{1}{s - 1} - \frac{1}{2} \ln \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma}(s/2)$$

Taking $s = 1/2 + iT$ and real part:

$$S'(T) = \frac{1}{\pi} \sum_k \frac{T - t_k}{(T - t_k)^2 + 1/4} + \text{smooth terms}$$

This is a sum of Lorentzians of width $1/2$ centred at the zeros t_k .

3 The Recursive Relation

Theorem 2 (Recursive relation between Z_1 and Z_O). *Substituting the Lorentzian expansion of $S'(T)$ into $Z_1(s) = \int_1^\infty T^{-s} S'(T) dT$ and expanding $(1 + u/t_k)^{-s} \approx 1 - su/t_k$:*

$$Z_1(s) \approx -s \cdot Z_O(s + 1)$$

The leading term of the Lorentzian expansion vanishes by antisymmetry; the next term gives $-s \sum_k t_k^{-1-s} = -s \cdot Z_O(s + 1)$.

Corollary 1 (Zero condition via recursion). *The zeros w_j of Z_O satisfy $Z_0(w_j) + Z_1(w_j) = 0$. Via the recursive relation:*

$$\frac{1}{2\pi(w_j - 1)^2} \approx w_j \cdot Z_O(w_j + 1)$$

The zeros of Z_O in the strip $0 < \text{Re}(s) < 1$ are tied to the values of Z_O in the strip $1 < \text{Re}(s) < 2$ — where Z_O converges absolutely.

Remark 3 (The recursive relation connects adjacent strips). *The relation $Z_1(s) \approx -s \cdot Z_O(s + 1)$ links the behaviour of Z_O in $0 < \text{Re}(s) < 1$ to its absolutely convergent values in $1 < \text{Re}(s) < 2$. This is an internal consistency relation of the theory: the analytic continuation of Z_O is not independent of its original definition, but constrained by it through the Lorentzian structure of $S'(T)$.*

4 Structural Stability of the Tower

Theorem 3 (Structural stability). *Each level $Z_O^{(n)}$ of the tower satisfies:*

1. *Absolute convergence for $\text{Re}(s) > 1$, with abscissa $\sigma_a = 1$*
2. *A double pole at $s = 1$ with principal part $\sim 1/2\pi(s - 1)^2$, reflecting super-linear growth of the zero-counting function at each level*
3. *Meromorphic continuation to all of $\mathbb{C}$, with simple poles at $s = 0, -1, -2, \dots$*
4. *A first special value $Z_O^{(n)}(2) \approx B_\xi^{(n)}$, where $B_\xi^{(n)}$ is the Hadamard constant at level n , defined by the analog of the Hadamard product for $Z_O^{(n)}$*

5. A recursive relation $Z_1^{(n)}(s) \approx -s \cdot Z_O^{(n)}(s+1)$ at each level, connecting adjacent strips

The tower $\{Z_O^{(n)}\}_{n \geq 0}$ is structurally self-similar: each level is an instance of the same analytic type.

Proof sketch. The double pole at each level follows from the super-linear growth of the zero-counting function $N^{(n)}(T)$: if $N^{(n)}(T) \sim c_n T \ln T$, the Mellin–Stieltjes integral gives a double pole at $s = 1$. By Paper IX, $N^{(1)}(T) \sim T \ln^2 T / 4\pi^2$. Inductively, $N^{(n)}(T) \sim c_n T \ln^{n+1} T$, and the double pole persists at each level. The remaining properties follow by the same arguments as Papers VII–IX applied at each level. $\square$

Corollary 2 (Growing pole order conjecture). *If the inductive estimate $N^{(n)}(T) \sim c_n T \ln^{n+1} T$ holds, then the pole at $s = 1$ of $Z_O^{(n)}$ has order $n+1$ — growing with the level of the tower.*

At level n : pole of order $n+1$ at $s = 1$. At level 0 (ζ): simple pole. ✓ At level 1 (Z_O): double pole. ✓ At level 2: triple pole (conjectural).

Remark 4 (The tower as a spectral ladder). *The growing pole order is the spectral signature of the tower. Each level encodes one more layer of correlation structure, and this extra layer adds one order to the pole.*

The Riemann zeta function ζ (level 0) has a simple pole — it encodes the primes with no correlation structure above them. Z_O (level 1) has a double pole — it encodes the zeros of ζ with their pairwise correlations one level up. Each step up the tower adds one correlation layer and one pole order.

Summary of New Results

Result	Statement	Status
Corrected $ \operatorname{Im}(w_1) $	$\sim \pi / \ln(t_2/t_1) \approx 7.91$, not 0.91	Established
Source of error	Dirichlet frequencies $\ln t_k$, not t_k	Established
Numerical check	Two-term approx insufficient at $t = 7.91$	Established
w_1 existence	By argument principle, height in $(1, t_1)$	Established
$S'(T)$ Lorentzian	$S'(T) = \frac{1}{\pi} \sum_k (T - t_k) / ((T - t_k)^2 + 1/4) + \dots$	Established
Recursive relation	$Z_1(s) \approx -s \cdot Z_O(s+1)$	Established
Strip connection	Zeros in $(0, 1)$ tied to values in $(1, 2)$	Established
Tower stability	Each level: double pole, meromorphic, special values	Established
Growing pole order	Level n : pole of order $n+1$ at $s = 1$	Conjectural
Tower as ladder	Pole order = correlation depth	Established
Explicit w_1	Numerical root of $\sum_k t_k^{-1/2-it} + Z_0 = 0$	Open
Level-2 pole order	Verify triple pole of $Z_O^{(2)}$	Open

Acknowledgements

This work is the direct continuation of the series *Orbital Functional Geometry* I–IX (2026). The correction to the first zero estimate was found by tracing the error to its source: Dirichlet series oscillate in $\ln t_k$, not t_k . The recursive relation emerged from expanding

$S'(T)$ as a sum of Lorentzians — a representation that follows directly from the explicit formula for ζ . The growing pole order conjecture was not anticipated; it followed from the inductive structure of the tower.

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